- Cours pour les absents -

Comportement temporel des S.L.C.I.

I. Systèmes du 1er ordre :

$$F(p) = \frac{K}{1+\tau p}$$

K : gain statique.

τ : constante de temps.

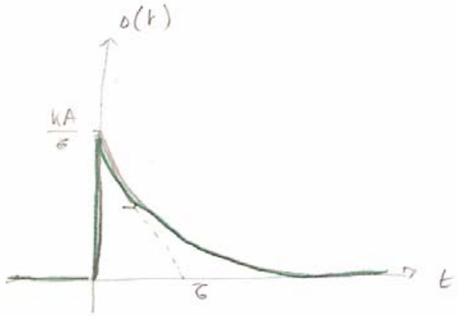
1. Réponse à une impulsion.

$e(t) = A\,\delta(t)$ $\quad E(p) = A$.

$$S(p) = F(p)\,E(p) = \frac{KA}{1+\tau p}$$

$$e^{-at} \rightarrow \frac{1}{p+a}$$

$$\boxed{t \geq 0 \quad s(t) = \frac{KA}{\tau} e^{-\frac{t}{\tau}}}$$



$$\lim_{t \to +\infty} s(t) = 0$$

$$\lim_{t \to 0^+} s(t) = \frac{KA}{\tau}$$

$$s'(t) = -\frac{KA}{\tau^2} e^{-\frac{t}{\tau}}$$

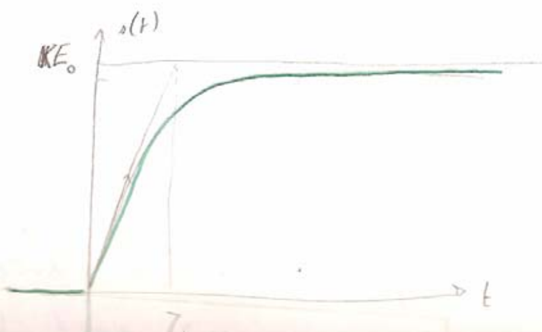
2. Réponse à un échelon :

$e(t) = E_0\, u(t)$ $\quad E(p) = \frac{E_0}{p}$

$$S(p) = \frac{KE_0}{p(1+\tau p)} \qquad S(p) = KE_0\left(\frac{A}{p} + \frac{B}{1+\tau p}\right)$$

$$A = \lim_{p \to 0} \frac{S(p)}{KE_0} \times p = 1$$

$$B = \lim_{p \to -\frac{1}{\tau}} \frac{S(p)}{KE_0}(1+\tau p) = -\tau.$$



$$S(p) = KE_0\left(\frac{1}{p} - \frac{\tau}{1+\tau p}\right)$$

$$\boxed{t \geq 0 \quad s(t) = KE_0\left(1 - e^{-\frac{t}{\tau}}\right)}$$

$$\lim_{t \to +\infty} s(t) = KE_0 .$$

Si $e(t)$ et $s(t)$ sont de même dimension :

$$\varepsilon_0 = \lim_{t \to +\infty} (e(t) - s(t)) = E_0(1-K)$$

le temps de réponse $t_{r\,5\%}$ est donné par :

$$s(t_{r\,5\%}) = 95\,\%\ KE_0$$

$$KE_0 = \left(1 - e^{-\frac{t_{r\,5\%}}{\tau}}\right) = 95\,\%\ KE_0$$

$$e^{-\frac{t_{r\,5\%}}{\tau}} = 5\,\%.$$

$$-\frac{t_{r\,5\%}}{\tau} \simeq -3$$

$$t_{r\,5\%} \simeq 3\tau.$$

NB : $s(5\tau) = 99\,\%\ s(\infty)$

$s(\tau) = 63\,\%\ s(\infty)$

3. Réponse à une rampe :

$e(t) = Vt\,u(t)$ $\qquad E(p) = \frac{V}{p^2}$

$$S(p) = \frac{KV}{p^2(1+\tau p)} = KV\left(\frac{A}{p^2} + \frac{B}{p} + \frac{C}{1+\tau p}\right)$$

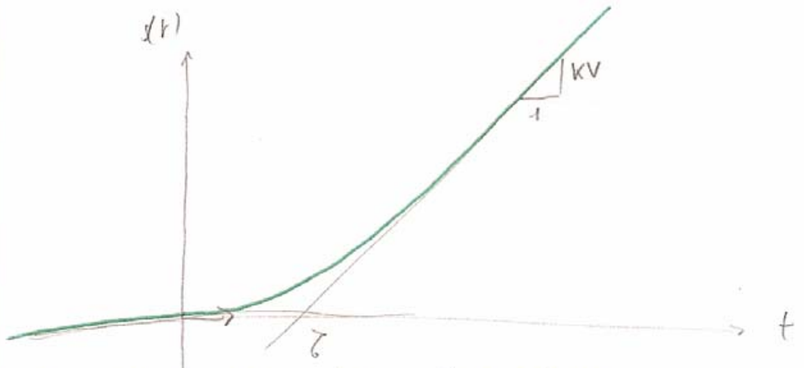
$$A = \lim_{p \to 0} \frac{S(p)}{KV} p^2 = 1$$

$$C = \lim_{p \to -\frac{1}{\tau}} \frac{S(p)}{KV}(1+\tau p) = \tau^2$$

En p^1 : $A\tau + B = 0$ $\quad B = -\tau$

$$S(p) = KV\left(\frac{1}{p^2} - \frac{\tau}{p} + \frac{\tau^2}{1+\tau p}\right)$$

$$\boxed{t \geq 0 \quad s(t) = KV\left(-t - \tau + \tau e^{-\frac{t}{\tau}}\right)}$$



Si $e(t)$ et $s(t)$ sont de même dimension.

$$\varepsilon_v = \lim_{t \to +\infty} \left(e(t) - s(t) \right) = \lim_{t \to +\infty} \left(v\,t - KV(t-\tau) \right)$$

si $K < 1$ $\quad \varepsilon_v \to +\infty$

si $K = 1$ $\quad \varepsilon_v \to V\tau$

si $K > 1$ $\quad \varepsilon_v \to -\infty$

II. Réponse d'un 2nd ordre à un échelon.

$$F(p) = \frac{K}{1 + \frac{2\xi}{\omega_0} p + \frac{1}{\omega_0^2} p^2}$$

K : gain statique

ω_0 : pulsation du système non amorti

ξ : facteur d'amortissement.

$e(t) = E_0 u(t) \qquad E(p) = \frac{E_0}{p}$

$$S(p) = \frac{KE_0}{p\left(1 + \frac{2\xi}{\omega_0} p + \frac{1}{\omega_0^2} p^2\right)} = \frac{KE_0 \omega_0^2}{p(\omega_0^2 + 2\xi\omega_0 p + p^2)}$$

$$\omega_0^2 + 2\xi\omega_0 p + p^2 = 0$$

$$\Delta = 4(\xi\omega_0)^2 - 4\omega_0^2 = 4\omega_0^2(\xi^2 - 1)$$

- $\xi > 1 \to$ 2 racines réelles $\quad p_{1,2} = \dfrac{-2\xi\omega_0 \pm 2\omega_0\sqrt{\xi^2 - 1}}{2}$
- $\xi = 1 \to$ 1 racine double. $\quad p_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$
  $p = -\omega_0$.
- $\xi < 1 \to$ 2 complexes conjugués.

$$p_{1,2} = -\xi\omega_0 \pm i\,\omega_0\sqrt{1 - \xi^2}.$$

1. $\xi > 1$ – Réponse apériodique.

$$p_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2-1}$$

$$s(p) = \frac{K E_0 \omega_0^2}{p(p-p_1)(p-p_2)}$$

On pose $\tau_1 = -\frac{1}{p_1}$ (1) $\quad \tau_2 = -\frac{1}{p_2}$ (2).

$$S(p) = \frac{K E_0 \omega_0^2}{p_1 p_2\, p\,(1+\tau_1 p)(1+\tau_2 p)} = \frac{K_1 E_0}{p(1+\tau_1 p)(1+\tau_2 p)}$$

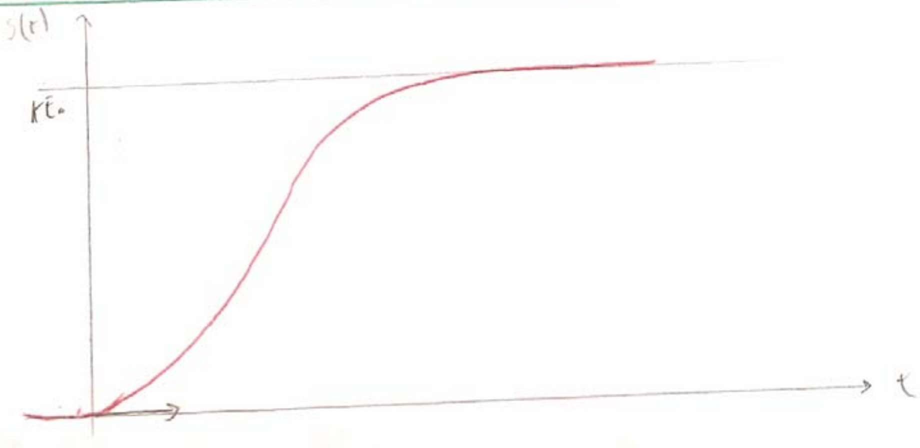
$$s(p) = K E_0\left(\frac{A}{p} + \frac{B}{1+\tau_1 p} + \frac{C}{1+\tau_2 p}\right)$$

$$A = \lim_{p \to 0} \frac{s(p)}{K E_0}\,p = 1$$

$$B = \lim_{p \to -\frac{1}{\tau_1}} \frac{s(p)}{K E_0}(1+\tau_1 p) = \frac{1}{\left(-\frac{1}{\tau_1}\right)\left(1-\frac{\tau_2}{\tau_1}\right)} = -\frac{\tau_1^2}{\tau_1-\tau_2}$$

$$C = \lim_{p \to -\frac{1}{\tau_2}} \frac{s(p)}{K E_0}(1+\tau_2 p) = -\frac{\tau_2^2}{\tau_2-\tau_1}$$

$$\boxed{t \geq 0 \quad s(t) = K E_0\left(1 - \frac{\tau_1}{\tau_1-\tau_2}e^{-\frac{t}{\tau_1}} - \frac{\tau_2}{\tau_2-\tau_1}e^{-\frac{t}{\tau_2}}\right)}$$



NB : $\tau_2 \gg \tau_1$

$\frac{\tau_1}{\tau_1 - \tau_2} \to 0$

$\frac{\tau_2}{\tau_2 - \tau_1} \to 1$

$s(t) = K E_0 \left(1 - e^{-\frac{t}{\tau_2}}\right)$

2. $\xi = 1$ - Réponse critique.

$p_{1,2} = -\omega_0$.

on pose $\tau = \frac{-1}{p_{1,2}} = \frac{1}{\omega_0}$

$S(p) = \frac{K E_0}{p(1+\tau p)^2}$

$S(p) = K E_0 \left(\frac{A}{p} + \frac{B}{(1+\tau p)^2} + \frac{C}{1+\tau p}\right)$

$A = \lim_{p \to 0} \frac{S(p)}{K E_0} p = 1$

$B = \lim_{p \to -\frac{1}{\tau}} \frac{S(p)}{K E_0} (1+\tau p)^2 = -\tau$

en p^1 $\quad 2A\tau + C = 0 \quad C = -\tau$

$S(p) = K E_0 \left(\frac{1}{p} - \frac{\tau}{(1+\tau p)^2} - \frac{\tau}{1+\tau p}\right)$

$t e^{-at} \to \frac{1}{(p+a)^2}$

$t \geq 0 \quad s(t) = K E_0 \left(1 - \frac{t}{\tau} e^{-\frac{t}{\tau}} - e^{-\frac{t}{\tau}}\right)$

$$\boxed{s(t) = K E_0 \left(1 - \left(\frac{t}{\tau} + 1\right) e^{-\frac{t}{\tau}}\right)}$$