Complément mathématique

Expression de $\overrightarrow{\text{grad}}U$ en coordonnées cartésiennes, cylindriques et sphériques

1 En coordonnées cartésiennes

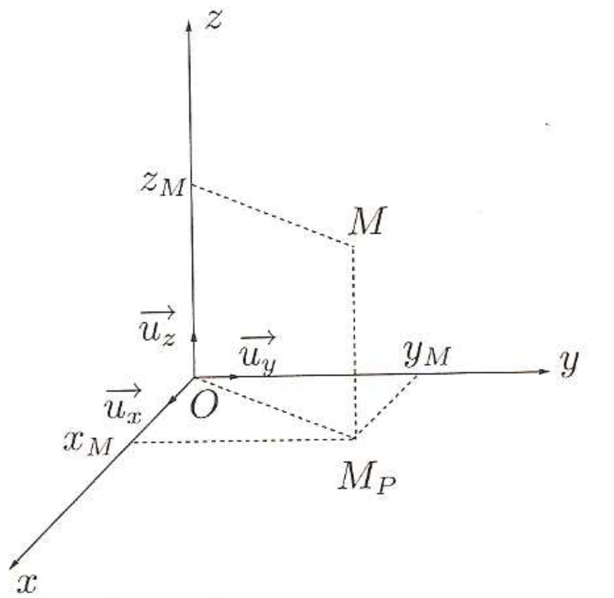

FIGURE 1 Coordonnées cartésiennes

On part de $dU(x,y,z) = \frac{\partial U(x,y,z)}{\partial x}dx + \frac{\partial U(x,y,z)}{\partial y}dy + \frac{\partial U(x,y,z)}{\partial z}dz$,
$\vec{V} = \overrightarrow{grad}U(x,y,z) = V_x\overrightarrow{u_x} + V_y\overrightarrow{u_y} + V_z\overrightarrow{u_z}$,
$\vec{V}.\overrightarrow{dl} = \overrightarrow{grad}U(x,y,z).\overrightarrow{dl} \equiv dU$,
$\overrightarrow{dl} = dx\overrightarrow{u_x} + dy\overrightarrow{u_y} + dz\overrightarrow{u_z}$.
Par identification, on trouve : $V_x = \frac{\partial U(x,y,z)}{\partial x}$, d'où :

$$\overrightarrow{\boldsymbol{grad}}\boldsymbol{U}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \frac{\partial \boldsymbol{U}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial \boldsymbol{x}}\overrightarrow{\boldsymbol{u_x}} + \frac{\partial \boldsymbol{U}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial \boldsymbol{y}}\overrightarrow{\boldsymbol{u_y}} + \frac{\partial \boldsymbol{U}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial \boldsymbol{z}}\overrightarrow{\boldsymbol{u_z}}$$

2 En coordonnées cylindriques

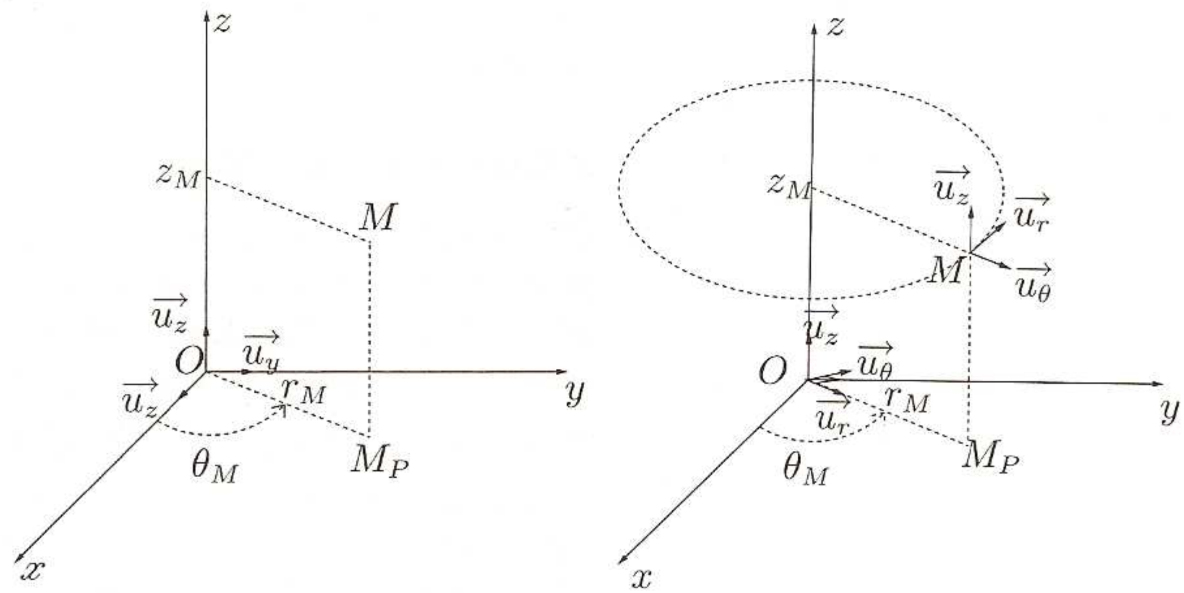

FIGURE 2 Coordonnées cylindriques (à gauche) et base locale cylindrique (à droite)

$\overrightarrow{dl} = dr\overrightarrow{u_r} + rd\theta\overrightarrow{u_\theta} + dz\overrightarrow{u_z}$,
$\vec{V} = V_r\overrightarrow{u_r} + V_\theta\overrightarrow{u_\theta} + V_z\overrightarrow{u_z}$,
puis $\vec{V}.\overrightarrow{dl} = dU(r,\theta,z) = \frac{\partial U(r,\theta,z)}{\partial r}dr + \frac{\partial U(r,\theta,z)}{\partial \theta}d\theta + \frac{\partial U(r,\theta,z)}{\partial z}dz$, d'où :

$$\overrightarrow{\boldsymbol{grad}}\boldsymbol{U}(\boldsymbol{r},\boldsymbol{\theta},\boldsymbol{z}) = \frac{\partial \boldsymbol{U}(\boldsymbol{r},\boldsymbol{\theta},\boldsymbol{z})}{\partial \boldsymbol{r}}\overrightarrow{\boldsymbol{u_r}} + \frac{\mathbf{1}}{\boldsymbol{r}}\frac{\partial \boldsymbol{U}(\boldsymbol{r},\boldsymbol{\theta},\boldsymbol{z})}{\partial \boldsymbol{\theta}}\overrightarrow{\boldsymbol{u_\theta}} + \frac{\partial \boldsymbol{U}(\boldsymbol{r},\boldsymbol{\theta},\boldsymbol{z})}{\partial \boldsymbol{z}}\overrightarrow{\boldsymbol{u_z}}$$

3 En coordonnées sphériques

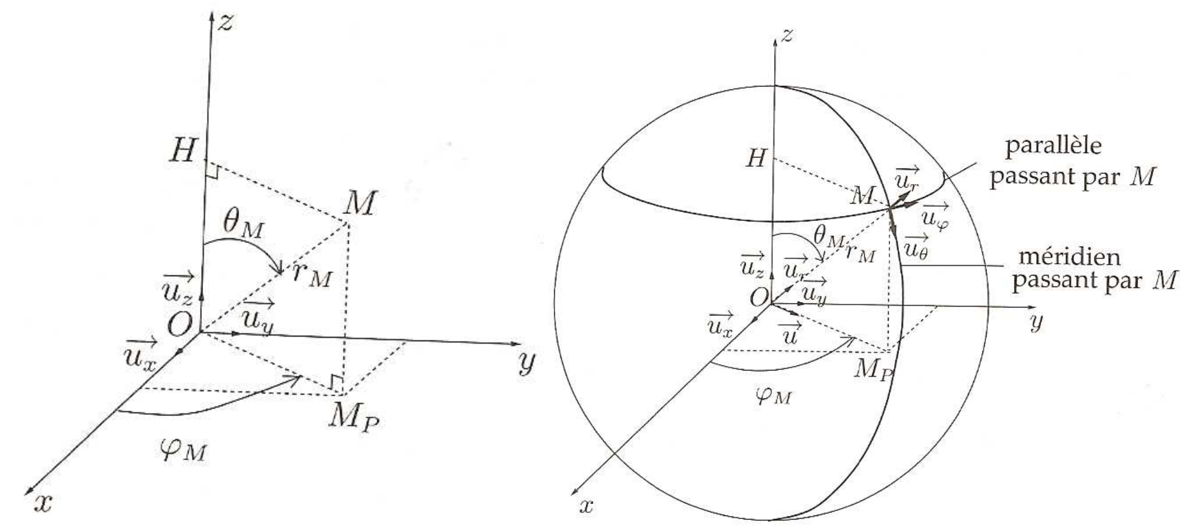

FIGURE 3 Coordonnées sphériques (à gauche) et base locale sphérique (à droite)

$\overrightarrow{dl} = dr\overrightarrow{u_r} + rd\theta\overrightarrow{u_\theta} + rsin\theta d\varphi\overrightarrow{u_\varphi}$,
$\vec{V} = V_r\overrightarrow{u_r} + V_\theta\overrightarrow{u_\theta} + V_\varphi\overrightarrow{u_\varphi}$,
puis $\vec{V}.\overrightarrow{dl} = dU(r,\theta,\varphi) = \frac{\partial U(r,\theta,\varphi)}{\partial r}dr + \frac{\partial U(r,\theta,\varphi)}{\partial \theta}d\theta + \frac{\partial U(r,\theta,\varphi)}{\partial \varphi}d\varphi$, d'où :

$$\boldsymbol{\overrightarrow{grad}U(r,\theta,\varphi) = \frac{\partial U(r,\theta,\varphi)}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial U(r,\theta,\varphi)}{\partial \theta}\overrightarrow{u_\theta} + \frac{1}{rsin\theta}\frac{\partial U(r,\theta,\varphi)}{\partial \varphi}\overrightarrow{u_\varphi}}$$